A RESISTANCE BOUND VIA AN ISOPERIMETRIC INEQUALITY

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An isoperimetric upper bound on the resistance is given. As a corollary we resolve two problems, regarding mean commute time on finite graphs and resistance on percolation clusters. Further conjectures are presented.

1. Introduction

It is natural and useful to interpret a graph as an electrical network by identifying the edges of the graph with one Ohm resistors. Then the effective resistance between vertices and sets admits a probabilistic and potential theoretic meaning, and is of interest, see for instance [6], [11], [9] for the standard background and definitions. In the next section we present a useful upper bound for the effective resistance between two vertices in the graph in terms of an isoperimetric quantity for connected sets containing one of these two vertices. In section 3 we bring two new applications.

2. A resistance bound

For a subset A of a graph G we denote by ∂A the *external* boundary, i.e. the vertices of $G \backslash A$ with neighbors in A. As usual, $\lfloor s \rfloor$ denotes the largest integer $\leq s$ and $\lceil s \rceil$ denotes the smallest integer $\geq s$. log denotes the logarithm to base 2.

Theorem 2.1. *Let G be a finite graph. Let w and u be vertices of G. Let $R_{w,u}$ be the electric resistance between w and u. Then*

$$(1) \quad R_{w,u} \le C(L_w + L_u), \quad L_v := \sum_{n=1}^{\lfloor \log |G| \rfloor} \max_{\substack{v \in A \\ A \text{ connected} \\ |G|2^{-(n+1)} < |A| \le |G|2^{-n}}} \frac{|A|}{|\partial A|^2} + \frac{1}{|\partial A|}$$

The $1/|\partial A|$ summands are relevant, of course, only in graphs with very high connectivity. For example, let G be a graph with $2n^2 + 2n + 2$ vertices, arranged as follows: $G = A_1 \cup \cdots \cup A_5$, $|A_1| = |A_5| = 1$, $|A_2| = |A_4| = n$ and $|A_3| = n^2$, and every vertex of A_i is connected to every vertex of A_{i+1} and to every vertex of A_{i-1}. It is easy to see that for this graph the resistance between A_1 and A_5 is $O(n^{-1})$, but

$$\sum \max \frac{|A|}{|\partial A|^2} = O(n^{-2}) \,.$$

To understand the conditions better it is worthwhile to examine the following silly example: Let G contain two components of equal size with w in one and u in the other. In this case the resistance is infinite. To get ∞ on the right hand side of (1) you need to have a set A with $|\partial A| = 0$ and the only such set is the complete half. Therefore replacing $|A| \le |G|2^{-n}$ with $|A| \le |G|2^{-n} - 1$ would render the lemma incorrect with any constant C.

Proof. We may assume that w and u are in the same component of G since otherwise both sides of (1) are infinite. We may assume that G has no other components and pay a price of 2 in the constant in (1). Connect a battery to w and u so that the voltage V satisfies $V(w) = 1$ and $V(u) = 0$. Denote the electric current by I. Denote by A_m a set of the m vertices with lowest voltage. If there are a number of possibilities (because some vertices have equal voltage), we choose A_m to be connected, which can always be done due to the maximum principle. Let $\theta(m) = \max_{v \in A_m} V(v)$. Denote

$$r_n := \min\{|\partial A| : u \in A, \ A \text{ connected and } |G|2^{-(n+1)} < |A| \le |G|2^{-n}\} \,.$$

Let $|G|2^{-(n+1)} < m \le |G|2^{-n}$. As already remarked, the set A_m is connected due to the maximum principle, contains u and therefore $|\partial A_m| \ge r_n$. For every vertex of ∂A_m examine the sum of the currents through all edges connecting it to A_m which we will call for simplicity the current through the vertex. Since the average current going through every vertex of ∂A_m is $I/|\partial A_m|$ we get that for at least $\frac{1}{2}r_n$ vertices the current through each is

$$\le 2\frac{I}{|\partial A_m|} \le \frac{2I}{r_n} \,.$$

Examine one such vertex v, and take an edge connecting v to A_m. The current through this edge, which is the difference of voltages, is $\leq 2I/r_n$. We get at least $\frac{1}{2}r_n$ vertices v where $V(v) \leq \theta(m) + \frac{2I}{r_n}$. This gives that $\theta(m + \left\lceil \frac{1}{2}r_n \right\rceil) \leq \theta(m) + \frac{2I}{r_n}$. Applying this inequality to $m_i = \left\lfloor |G| 2^{-(n+1)} \right\rfloor + 1 + i \left\lceil \frac{1}{2}r_n \right\rceil$ for all i from 0 up until the first i such that $m_{i+1} > |G| 2^{-n}$ and summing one gets after a simple calculation

$$\theta\left(\lfloor |G| 2^{-n} \rfloor + 1\right) - \theta\left(\left\lfloor |G| 2^{-(n+1)} \right\rfloor + 1\right) \leq \frac{2I}{r_n} \left(\frac{2 \left\lceil |G| 2^{-(n+1)} \right\rceil}{r_n} + 1 \right)$$

which we sum over n and get

$$\theta(\lfloor |G|/2 \rfloor + 1) - \theta(1) \leq CIL_u.$$

Since $\theta(1) = 0$ we are done with the neighborhood of u. An identical calculation around w will show

$$1 - \theta'(\lfloor |G|/2 \rfloor + 1) \leq CIL_w$$

where $\theta'(m) = \min_{v \in A'_m} V(m)$ and A'_m is a set of m vertices with maximal voltage which is a connected neighborhood of w. Since $A_{\lfloor |G|/2 \rfloor + 1}$ and $A'_{\lfloor |G|/2 \rfloor + 1}$ intersect we get $\theta(\lfloor |G|/2 \rfloor + 1) \geq \theta'(\lfloor |G|/2 \rfloor + 1)$ and then

$$1 \leq CI(L_u + L_w)$$

which finishes the proof.   ∎

3. Applications

3.1. Mean commute time

Let $\tau^* = \max_{v,u}(E_v T_u + E_u T_v)$. Where $E_v T_u$ is the expected hitting time for a random walk starting at v to hit u. By hitting time we mean here continuous hitting time, i.e. one puts on every edge an alarm clock with the ringing time distributed like an exponential variable and then move from a vertex along the first edge that rings, at the time it rings. τ^* is the maximal mean commute time.

Open problem 20 in chapter 6 of [1] asserts the following:

Show that for real $1/2 < \gamma < 1$ and $\delta > 0$, there exists a constant $K_{\gamma,\delta}$ with the following property. Let G be a regular n-vertex graph such that, for any subset A of vertices with $|A| \leq n/2$, there exist at least $\delta|A|^\gamma$ edges between A and A^c. Then $\tau^* \leq K_{\gamma,\delta} n$.

Theorem 2.1 allows to prove this under the assumption that the graph degree is bounded, in which case there is no difference between an isoperimetric condition phrased in terms of the number of edges (as in the problem) or in terms of the number of vertices (as in theorem 2.1). In this case we use the fact that the mean commute time between any two vertices u, w for simple random walk on a connected graphs equals $R_{u,w}|G|$, see chapter 4 of [1]. The isoperimetric condition gives in theorem 2.1 a bounded sum and the answer is positive. Note that we have not used regularity at all.

When one removes the assumption of bounded degree, the answer is negative. For example, take a graph with n vertices arranged in a circle such that each two neighbors are connected by $\lceil n^{2/3} \rceil$ edges. Then clearly the assumptions hold (with $\gamma = \frac{2}{3}$ and $\delta = 1$) but the conclusion fails as $\tau^* > c n^{4/3}$. It is not difficult to construct such an example with no multiple edges.

Theorem 2.1 can be used in the context of the Gaussian random field.

Remark. Bounding the height of Gaussian random field. The Gaussian field on a connected graph with n vertices is defined as follows. Given a finite connected graph $G = (V_G, E_G)$, for each directed edge $\overrightarrow{e} \in G$, let $Y_{\overrightarrow{e}}$ be a standard normal random variable, so that for opposite orientation of the same edge $Y_{\overleftarrow{e}} = -Y_{\overrightarrow{e}}$, and otherwise the variables $Y_{\overrightarrow{e}}$ are independent, and let $X_{\overrightarrow{e}}$ be $Y_{\overrightarrow{e}}$ Conditioned on the event that all sums of these variables, around a closed cycle in the graph, vanish. Pick a base vertex $v_0 \in G$, and define the height function on the vertices of the graph: $h(v_0) = 0$ and for any other $v \in G$ set $h(v) = \sum_{\gamma_v} X_e$, where γ_v is a path from v_0 to v in the graph. h is well defined, (See [8] or [4]). Janson [8] proved that $E|h(v) - h(u)|^2$ is equal to the resistance between v and u, when you replace each edge of G by a resistor of unit resistance. Theorem 2.1 then can be used to derive bounds on the height variance of the Gaussian random field as well. E.g. if $|\partial S| > C|S|^{1/2+\epsilon}$ for any $S \subset G, 0 < |S| < |G|/2$, then the height variance is bounded uniformly.

3.2. Resistance of the 2D supercritical percolation cluster

Consider supercritical $(p > 1/2)$ bond percolation on the $n \times n$ box of the 2D square lattice. Grimmett (private communication) asked: show that almost surely, with respect to the percolation measure $\mathbb{P}_p$, the maximal resistance between any pair of vertices on the giant component is bounded by $C \log n$. Denote by R_C^n the maximal resistance between any pair of vertices on the largest cluster of the percolation inside the $n \times n$ box. Indeed we have

Corollary 3.1.

$$\mathbb{P}_p(R_C^n < C_p \log n) \to 1.$$

Proof. By theorem 2.1 it is enough to show that for C sufficiently large and $c > 0$ sufficiently small the probability that any connected set S in the giant component of size bigger than $C \log n$, has boundary of size bigger than $c|S|^{1/2}$ goes to 1 with n. This indeed follows from an old argument of Kesten [7] and is done explicitly in section 2.3 of [3]. ∎

For more on the relationships between random walks and percolation clusters see [10], [3] and the references therein.

4. A Conjecture

The Cheeger constant of a finite *transitive* graph is at least the reciprocal of the diameter (see [2]). We hope the following stronger conjecture holds.

Conjecture 4.1. Let G be finite, connected and vertex transitive. Show that if $\operatorname{diam}(G) < |G|^\alpha$ then $|\partial S| > c_\alpha |S|^{1-\alpha}$ for any S, $1 \le |S| \le |G|/2$.

If true, the first part of the next conjecture will follow along the lines of proof of theorem 2.1.

Conjecture 4.2. Let G be finite, connected and vertex transitive. For any two vertices

$$R_{v,u} < C + \frac{\operatorname{diam}(G)^2 \log |G|}{|G|}.$$

In addition, if the diameter is $o(|G|)$ then the electric resistance between any two vertices is $o(\operatorname{diam}(G))$.

These conjectures should be compared with the case of infinite vertex transitive graphs which was settled by Varopoulos [13], the only recurrent vertex transitive graphs are roughly isometric to $\mathbb{Z}$ or $\mathbb{Z}^2$.

This might be the point to note that for vertex transitive graphs it is possible to prove isoperimetric inequalities of this kind by examining balls only, using a result of Coulhon and Saloff-Coste [5]. Their theorem (theorem 1 ibid.) is stated for Cayley graphs of finitely generated groups, but it is not difficult to generalize it to edge transitive graphs (or to vertex transitive graphs with bounded degree). A similar result can be found in [12].

References

[1] D. ALDOUS and J. FILL: *Reversible Markov Chains and Random Walks on Graphs*, http://stat-www.berkeley.edu/users/aldous/book.html

[2] L. BABAI and M. SZEGEDY: Local expansion of symmetrical graphs, *Combin. Probab. Comput.* **1(1)** (1992), 1–11.

[3] I. BENJAMINI and E. MOSSEL: On the mixing time of simple random walk on the super critical percolation cluster, *Prob. Theor. and Rel. Fields* **125(3)** (2003), 408–420.

[4] I. BENJAMINI and G. SCHECHTMAN: Upper bounds on the height difference of the Gaussian random field and the range of random graph homomorphisms into $\mathbb{Z}$, *Random Structures Algorithms* **17(1)** (2000), 20–25.

[5] T. COULHON and L. SALOFF-COSTE: Isoperimetrie pour les groupes et les varietes, *Rev. Mat. Iberoamericana* **9** (1993), 293–314. In French.

[6] P. DOYLE and J. SNELL: *Random Walks and Electric Networks*, http://front.math.ucdavis.edu/math.PR/0001057

[7] H. KESTEN: On the time constant and path length of first-passage percolation, *Adv. in Appl. Probab.* **12(4)** (1980), 848–863.

[8] S. JANSON: *Gaussian Hilbert spaces*, Cambridge (1998).

[9] R. LYONS with Y. PERES: *Probability on trees and networks*, http://www.math.gatech.edu/~rdlyons/prbtree/prbtree.html

[10] P. MATHIEU and E. REMY: Isoperimetry and heat kernel decay on percolation clusters, *C. R. Acad. Sci. Paris Sér. I Math.* **332(10)** (2001), 927–931.

[11] Y. PERES: *Probability on Trees: An Introductory Climb*, http://stat-www.berkeley.edu/~peres/

[12] L. SALOFF-COSTE: Isoperimetric inequalities and decay of interated kernels for almost-transitive Markov chains, *Combinatorics, Probability and Computing* **4** (1995), 419–442.

[13] N. VAROPOULOS, L. SALOFF-COSTE and T. COULHON: Analysis and geometry on groups, *Cambridge Tracts in Mathematics* **100**. Cambridge University Press, Cambridge (1992), xii+156 pp.

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